# Topology, normalisability and the Schrödinger equation: Compact QED (2+1).

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**Abstract**

For the special case of compact QED in (2+1) dimensions, we calculate the non-Gaussian vacuum wave-functional to second order in the monopole fugacity and obtain the effective photon mass. Our method presents some hope for understanding the connection between variational and systematic approaches to understanding the non-perturbative wave-functional.

# 1 Introduction

In recent years there has been a revival of interest in Hamiltonian methods for Yang-Mills gauge theories. These studies have for the most part fallen into two groups: one group tries to solve the functional Schrödinger equation in some sort of systematic expansion [1]-[9], the other makes use of a Gaussian ansatz for the wave-functional, and a variational principle that minimises the vacuum energy on the Gaussian state [10]-[14]. Both of these approaches have led to interesting insights into the vacuum structure of non-abelian gauge theories, but in both cases many important questions remain unanswered.

Systematic methods run into a number of technical difficulties that have been resolved in some cases but not in general. One of the first problems is to rewrite the Hamiltonian in terms of suitable gauge-invariant variables. This must be done in such a way that the relevant systematic expansion is well-defined and renormalisable. Certain expectations, such as the existence of a derivative expansion of the wave-functional at large distances [15] are frustrated by complications arising from the existence of topologically non-trivial classical field configurations [7, 16]; some choices of gauge-invariant variables can take them into account implicitly, or one can perform an explicit expansion around classical solutions with non-trivial toplogy. Dynamically generated mass parameters emerge from the analysis in various ways [7, 11].

The variational approach sidesteps many of these issues, at the cost of restricting to a (gauge-projected) Gaussian state. This allows many concrete calculations to be performed; on the other hand, though it is reasonable to expect that the Gaussian state captures the important physics, this is difficult to justify. It would be desirable to "elevate" the results of the variational method to the full non-Gaussian theory, or at least to understand the relation between the two.

In this paper we will achieve this for the simple case of compact QED in (2+1) dimensions, extending the analysis of [11] to the full non-Gaussian theory. This model exhibits many interesting features such as dynamical mass generation and confinement, and though the mechanisms for these features are special to (2+1) dimensions, many general features of our method are more widely applicable. Our analysis makes use of a systematic expansion in the monopole fugacity, which is closely related to the dynamically generated photon mass. Although the dynamically generated mass is of the order of the monopole fugacity, to calculate it in the non-Gaussian theory requires an analysis to second order in this parameter. Results to higher order are obtainable in principle.

The variational principle is essentially trivial, and the meat of the calculation is in the effective monopole dynamics, but the former nevertheless allows us to determine the contributions of higher $n$-point functions to the propagator and solve the functional Schrödinger equation order by order. Normalisability conditions on the wave-functional also play a role.

The ultimate aim of our investigations is to make progress towards Hamiltonian methods for the study of QCD and Yang-Mills in (3+1) dimensions. It is our belief that though such methods are currently rudimentary, they can be developed to the point where such methods can rival the explicit calculations available, for example, in lattice gauge theory.

## 2 Free QED

To begin with consider free non-compact electrodynamics in (2+1) dimensions. The Hamiltonian is

$$H = \tfrac{1}{2}(B^2 + E_i^2), \tag{1}$$

where $B = \epsilon_{ij}\partial_i A_j$ and the electric field operator is represented as

$$E_i \sim -i\frac{\delta}{\delta A_i}. \tag{2}$$

If we make a Gaussian ansatz for the vacuum wave-functional

$$\Psi[B] = \exp\{-\tfrac{1}{2}BCB\}, \tag{3}$$

(here we have made use of a matrix notation, so that $BCB = \int d^2x d^2y B(x)C(x-y)B(y)$, etc.) then the Schrödinger equation $H\Psi = E\Psi$ implies that (in momentum space)

$$-p^2C(p)^2 + 1 = 0, \qquad E/V = \frac{1}{2}\int \frac{d^2p}{4\pi^2}p^2C(p). \tag{4}$$

Here $E$ is the vacuum energy and $V$ is the volume of space. From (4) we immediately obtain $C(p) = 1/p$.

We can also derive this result by means of a variational principle: according to the standard prescription for calculating expectation values we have

$$\langle B^2\rangle = \frac{\int DA_i B^2\Psi[B]\Psi^*[B]}{\int DA_i\Psi[B]\Psi^*[B]} = \frac{1}{2}\int \frac{d^2p}{4\pi^2}\frac{1}{C(p)}, \tag{5}$$

while

$$\langle E_i^2\rangle = \frac{\int DA_i E_i^2\Psi[B]\Psi^*[B]}{\int DA_i\Psi[B]\Psi^*[B]} = \frac{1}{2}\int \frac{d^2p}{4\pi^2}p^2C(p). \tag{6}$$

Minimising the expectation value of the Hamiltonian then gives $C(p) = 1/p$ as before (and the virial theorem for harmonic oscillators is satisfied since $\langle E_i^2\rangle = \langle B^2\rangle$).

There is a third way of deriving this result that does not depend on making a Gaussian ansatz. If we assume a more general form for the vacuum wave-functional[1]

$$\Psi[B] = \exp\{-\tfrac{1}{2}BCB - \tfrac{1}{4!}C_4B^4 - \tfrac{1}{6!}C_6B^6 - \ldots\}, \tag{7}$$

then the Schrödinger equation becomes

$$\begin{aligned} E/V &= \frac{1}{2}\int \frac{d^2p}{4\pi^2}p^2C(p) \\ 0 &= -C(p)^2p^2 + \tfrac{1}{2}C_4(p)p^2 + 1 \\ &\vdots \end{aligned} \tag{8}$$

[1]In matrix notation $C_4B^4 = \int d^2x_1 \ldots d^2x_4 C_4(x_1,x_2,x_3,x_4)B(x_1)B(x_2)B(x_3)B(x_4)$, etc.

In (8) $C_4(p)$ is the Fourier transform of $C_4(x) = \int d^2y C_4(0, y, y, x)$, and we have assumed translational invariance. The expression for the vacuum energy density is unchanged, while the order $B^2$ part of the Schrödinger equation acquires a contribution from the four-point function. The vertical dots indicate the existence of further equations at order $B^4$ and so on that will not concern us much for the moment.

In addition to the equations (8), the vacuum wavefunctional must satisfy normalisability conditions $C \geq 0$, $C_4 \geq 0$, etc. If we minimise the vacuum energy density subject to these normalisability conditions we get $C(p) = 1/p$ as before, while $C_4$ and all other n-point functions vanish. Note that the energy minimisation determines the contribution of the 4-point function to the 2-point function, allowing us to solve (8) order by order in $B^2$. This is a general feature of our methods.

# 3 Compact QED

The compact theory differs from the non-compact one by the presence of Dirac monopoles, which play the role of instantons in three space-time dimensions. As usual, we can represent the vacuum as a sum over instanton sectors

$$\Psi[B] = \sum_n \Psi_n[B], \tag{9}$$

where $\Psi_n$ represents the vacuum in the n-monopole sector. Equivalently, we can require the vacuum to be invariant under the action of the vortex creation operator

$$V(x) = \exp\left\{\frac{i}{g}\int d^2y \frac{\epsilon_{ij}(x-y)_j}{(x-y)^2} E_i(y)\right\}, \tag{10}$$

so that the vacuum wave-functional is a superposition of wave-functionals of the non-compact theory with an arbitrary number of vortices and anti-vortices at any spatial point [11]:

$$\tilde{\Psi}[B] = \frac{1}{n_+!n_-!} \sum_{n_+,n_-=0}^{\infty} \prod_{\alpha=1}^{n_+} \int d^2x_\alpha \Lambda^2 V(x_\alpha) \prod_{\beta=1}^{n_-} \int d^2x_\beta \Lambda^2 V^*(x_\beta) \Psi[B]. \tag{11}$$

We have introduced an ultraviolet cutoff $\Lambda$. If the theory is regarded as the unbroken sector of the Georgi-Glashow model, then $\Lambda$ is related to the charged vector boson mass $M_W$; the details depend on the UV dynamics [17]. The reason for the aforementioned equivalence is as follows: inner-products of the form (5) are regularised by point-splitting, and monopoles that live on the quantisation surface have Dirac lines that can intersect just one of the copies of the wave-functional in the inner-product, inducing a vortex, or neither. Thus the vacuum, summed over monopole sectors, is invariant under multiplication by the vortex operator.

The vortex/monopole is represented on the wave-functional by a shift of $\frac{2\pi}{g}\delta^2(x)$ in the magnetic field $B(x)$. Introducing the monopole density $\rho(x) = \frac{2\pi}{g}\sum_{\alpha,\beta}(\delta^2(x-x_\alpha) - \delta^2(x-x_\beta))$, the sum over monopoles in the non-Gaussian wave-functional (7) gives

$$\tilde{\Psi}[B] = Z^{-1} \sum_{\rho} \exp\{-\tfrac{1}{2}(B+\rho)C(B+\rho) - \tfrac{1}{4!}C_4(B+\rho)^4 - \tfrac{1}{6!}C_6(B+\rho)^6 - \ldots\}. \quad (12)$$

We have normalised the wave-functional to $\tilde{\Psi}[0] = 1$ with the factor $Z = \sum_\rho \Psi[\rho]$. The vacuum energy density is then

$$E/V = \frac{1}{2} Z^{-1} \sum_{\rho} \int \frac{d^2p}{4\pi^2} \left[ p^2 C(p) + [\tfrac{1}{2} p^2 \rho^2 C_4(p) - p^2 (\rho C(p))^2\}] + O(\rho^4) \right] \Psi[\rho], \quad (13)$$

where $\rho C(x) = \int d^2y \rho(y) C(x-y)$, etc.

Expectation values of the monopole density are often calculated in the effective low energy theory, which takes the form of a sine-Gordon theory [11, 17]. They can also be calculated directly by expanding the wave-functional in monopole sectors; thus

$$Z = \sum_{\rho} \Psi[\rho] = 1 + 2Vz + \ldots \quad (14)$$

where $z = \Lambda^2 \exp\{-\frac{2\pi^2}{g^2} C(0) - \frac{2\pi^4}{3g^4} C_4(0,0,0,0) - \ldots\}$ is the monopole fugacity, and we have included only the dominant one (anti-)monopole sector explicitly. Note that the expression for the monopole fugacity is to be understood in a UV regularised sense, so that $C(0) \sim C(1/\Lambda)$, etc. It can be shown that all volume factors cancel in the expression (12) (logarithmic divergences appear in the wave-functional at higher orders, and can be removed by normal-ordering). This kind of structure is familiar in mass perturbation theory [18, 19].

To first order in $z$ we have

$$\langle \rho(p_1) \ldots \rho(p_n) \rangle = Z^{-1} \sum_{\rho} \rho(p_1) \ldots \rho(p_n) \Psi[\rho] = \frac{8\pi^2}{g^2} z \delta^2(p_1 + \ldots + p_n). \quad (15)$$

Now to this order the expression (13) for the energy density can be simplified by using the equations (8). We do not need to consider $O(z)$ corrections to these equations, as they will lead to $O(z^2)$ corrections to the vacuum energy. The order $\rho^2$ terms in (13) give an order $z$ term, and the order $\rho^4$ and higher terms cancel, giving

$$E/V = \frac{1}{2} Z^{-1} \sum_{\rho} \int \frac{d^2p}{4\pi^2} \left[ p^2 C(p) - \frac{8\pi^2}{g^2} z \right] \Psi[\rho]. \quad (16)$$

This is the same result we would have obtained if we had assumed that the wave-functional tends smoothly to the wave-functional of the non-compact theory as $z \to 0$, so that at order 1, $C(p) = 1/p$ and $C_4(p) = 0$, etc.

To proceed further it is useful to expand the logarithm of the wave-functional in powers of the magnetic field. So we write

$$\ln \Psi \;\; = \;\; \ln \Big( e^{-\frac{1}{2} BCB - \frac{1}{4!} C_4 B^4 - \ldots}$$

$$\times \left(1 + z\int d^2x_0 \sum_{\pm} e^{\pm\frac{2\pi}{g}\int d^2x C(x_0-x)B(x) + \frac{2\pi^2}{g^2}\int d^2x d^2y C_4(x_0,x_0,x,y)B(x)B(y)+\ldots}\right)\Bigg)$$

$$= -\tfrac{1}{2}BCB + z\tfrac{4\pi^2}{g^2}\left[C^2B^2 - \tfrac{1}{2}C_4B^2\right] + \ldots$$

$$= -\tfrac{1}{2}BC^{eff}B + \tfrac{1}{4!}C_4^{eff}B^4 + O(B^6) + O(z^2). \qquad (17)$$

Here the sum over $\pm$ corresponds to the one-monopole/one anti-monopole sectors and we have exactly the same simplification that we observed in the vacuum energy, giving a shift $C(p) \to C^{eff}(p) = C(p) - \frac{8\pi^2 z}{g^2p^2}$. For higher orders in $B$ we find

$$C_{2n}^{eff}(p_1,\ldots,p_{2n}) = C_{2n} - \frac{\delta(p_1+\ldots+p_{2n})}{p_1\ldots p_{2n}}\frac{2(2\pi)^{2n}z}{(2n-2)!g^{2n}}. \qquad (18)$$

Writing the equations (8) in terms of $C^{eff}$, $C_4^{eff}$, etc. we have (to first order in the monopole fugacity) the same equations and normalisability conditions as before, and hence the same solution $C^{eff}(p) = 1/p$, $C_{2n}^{eff} = 0$ for $n > 1$. Thus $C(p) = \frac{1}{p} + \frac{8\pi^2 z}{g^2p^2}$, etc.

Our failure to see mass generation at this order does not come as so much of a surprise if we re-examine the variational Gaussian calculation of [11]. There the order $z \sim m^2$ contributions from the electric and magnetic energy densities cancel against one another. To see the mass generation in our formalism, we will have to extend our analysis to order $z^2$.

# 4 The second-order calculation

If we include two (anti-)monopole configurations in the calculations of the last section, we find the two monopole contribution

$$\langle\rho(x)\rho(y)\rangle_2 = z^2\left(\pm e^{\pm C(x-y)} + \delta^2(x-y)\int_{w>L} d^2w\left(e^{\pm C(w)} - 1\right)\right). \qquad (19)$$

Here the sign is $\pm$ as the monopoles have equal or opposite charge, and in the second term we have a partial cancellation of the infrared divergence due to the normalisation factor $Z$, though a logarithmic divergence remains. The monopole density in the vacuum is low for $z \ll 1$ so that (11) is the partition function for a dilute gas, in which monopoles are assumed to be widely separated, hence the cutoff in the integral.

Putting this result into (17) we find the order $z^2$ contribution to $C^{eff}(x-y)$:

$$8\pi^2\frac{z^2}{g^2}\left\{\left[\int_{|x_1-x_2|>L} d^2x_1 d^2x_2 \frac{1}{|x-x_1|}\frac{1}{|x_2-y|}\sinh\left(\frac{1}{g^2|x_1-x_2|}\right)\right]\right.$$
$$\left. + \delta^2(x-y)\int_{w>L} d^2w\left(\cosh\left(\frac{1}{g^2w}\right) - 1\right)\right\}. \qquad (20)$$

In the dilute gas approximation we can take $\sinh\left(\frac{1}{g^2|x_1-x_2|}\right) \approx \frac{1}{g^2|x_1-x_2|}$ and evaluate the first integral in (20) as $\frac{|x-y|}{g^2}$. This corresponds to a mass term in the two-point function; the puzzle is that the mass $m^2 \sim \frac{z^2}{g^4}$ appears to be of order $z^2$, whereas we know from [17] that the effective photon mass is of order $z$.

Again, the problem is that the topological expansion makes it difficult to read the physical mass off from the wave-functional directly. What we need to determine the physical mass is a vacuum expectation value like $\langle B(x)B(y)\rangle$, which can be read off from the functional Fourier transform of the non-Gaussian wave-functional that we have constructed.

To obtain the functional Fourier transform we treat the four-point and higher functions as perturbations of the quadratic wave-functional. The Schrödinger equation again has the solution $C^{eff}(p) = 1/p$, $C^{eff}_{2n} = 0$ for $n > 1$, and we can write the wave-functional in terms of the field $A_i$ (with vector indices absorbed into the matrix notation) as

$$\Psi[A_i] = \exp\{-\tfrac{1}{2}A\tilde{C}A - \tfrac{1}{4!}\tilde{C}_4A^4 - \tfrac{1}{6!}\tilde{C}_6A^6 - \ldots\}, \tag{21}$$

where for example $\tilde{C}(p) = \mathcal{P}p^2C(p)$ with $\mathcal{P}$ a transverse projector. For our purposes we can now forget about the topological expansion, since VEV's of the magnetic field operator are unaffected by the presence of monopoles [11]. We are interested in the quadratic part of the Fourier transformed wave-functional, from which we can read off the physical mass.

The latter receives contributions from tadpole diagrams like $\int d^2x_1 \ldots d^2x_{2n}\tilde{C}_{2n}(x_1, \ldots, x_{2n})$ $\tilde{C}^{-1}(x - x_1)\tilde{C}^{-1}(x_2 - x_3)\ldots\tilde{C}^{-1}(x_{2n} - y)$, etc. which, when summed, give the final result for the photon mass

$$m^2 \sim \frac{z^2}{g^4}\exp\left(-\frac{\pi\Lambda}{2g^2}\right) = \frac{\Lambda^2}{g^4}z. \tag{22}$$

This result agrees with [11, 17], and is of first order in the monopole fugacity, as expected. Interestingly, the dynamical mass generation is given here by the condensation in the vacuum of a monopole/anti-monopole pair.

# 5 Conclusions

For the case of compact QED in (2+1) dimensions we have shown how the mass generation first demonstrated by Polyakov [17] and reproduced in the Hamiltonian formalism in [11] by means of a gauge-projected Gaussian ansatz, can be seen in the full non-Gaussian wave-functional. There is a variational principle implicit in any solution of the Schrödinger equation, and our method provides a link between variational methods based on a Gaussian ansatz, and attempts to solve the functional Schrödinger equation exactly in some non-perturbative expansion.

In our analysis the variational principle and normalisability conditions on the wave-functional played a role, but the mass generation was seen to be a feature of the effective monopole dynamics. In general, explicit expansion around topological solutions may be the best way to tackle a systematic Hamiltonian analysis of non-abelian gauge theories.

The extension of these functional Schrödinger methods to fermions and superfields is straightforward. The study of supersymmetric theories along these lines is strongly suggested by the non-renormalisation theorems afforded by such theories; these simplify the quantisation considerably, as well as facilitating the interpolation between different perturbative regimes.

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